Trigonometry made somewhat easier

To understand trigonometry, it is most useful to first understand two properties of triangles: the Pythagorean theorem and the relationships between similar triangles. To begin, we will review each of these.

Pythagorean theorem

Given a right-angle triangle with sides of length *A*, *B* and *C*, the last being on the hypotenuse (the line opposite the right angle), as shown in Figure 1, the Pythagorean theorem says that $A^2 + B^2 = C^2$.

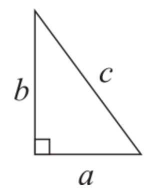


Figure 1. A right-angled triangle.

Proof:

Consider a square with sides *a* + *b*. This square has area $(a+b)^2 = a^2 + 2ab + b^2$. Inscribe in each of the corners of this square a triangle with sides *abc*, as shown in Figure 2.

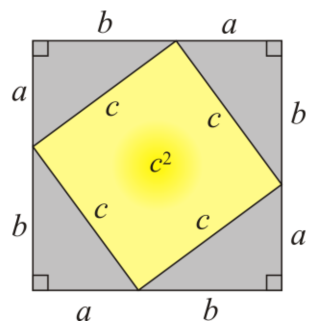


Figure 2. Proof of the Pythagorean theorem.

The central yellow square has sides equal to *c*, and thus the area of the central square is c^2. The area of each of the four triangles is $\frac{1}{2}ab$, and therefore the area of the central square is equal to the total area minus four times the area of each triangle:

$$c^2 = (a+b)^2 - 4\left(\frac{1}{2}ab\right) = a^2 + 2ab + b^2 - 2ab = a^2 + b^2 .$$

Thus, we have the identity that $c^2 = a^2 + b^2$. ■

As a visual proof, you can take the triangles within the square with sides $a + b$ and rearrange them. In both squares, the sum of the areas of the gray triangles is equal, and therefore the balance, too, must have the same area. In the left-hand square of Figure 3, we see that the central yellow square has area c^2, while in the right-hand square, we see that the area of the red and blue squares is $a^2 + b^2$.

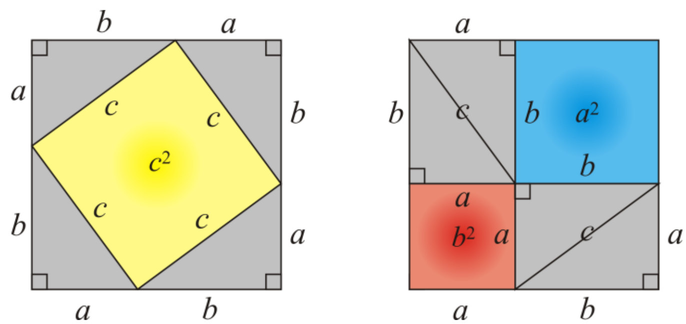


Figure 3. Alternate visual proof of the Pythagorean theorem.

Thus, we have the identity that $c^2 = a^2 + b^2$. ■

Properties of similar triangles

Given two similar triangles (with the equal corresponding angles) as shown in Figure 4, the properties of similar triangles gives us that the rations between corresponding sides are equal; that is, $\frac{a}{b} = \frac{d}{e}$, $\frac{a}{c} = \frac{d}{f}$ and $\frac{b}{c} = \frac{e}{f}$.

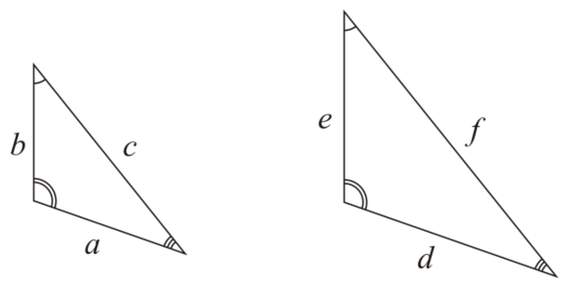


Figure 4. Two similar triangles.

We can now start with trigonometry. Most of you will remember the geometric interpretation of $\sin(\theta)$ and $\cos(\theta)$, as shown in Figure 5: given a unit circle (a circle of radius 1) centered at the origin, a line extended from the origin forming an angle θ with the right horizontal axis intersects a unit circle at the point $(\sin(\theta), \cos(\theta))$.

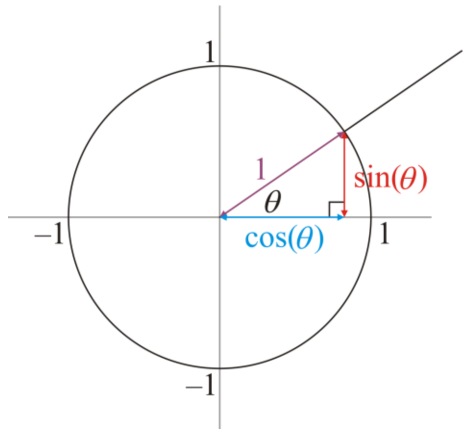
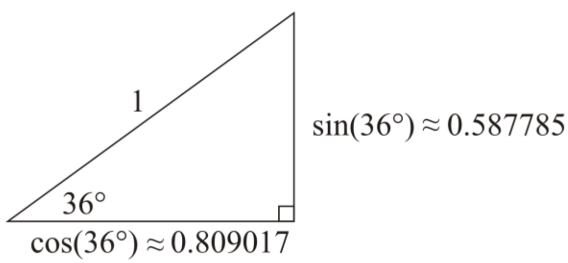
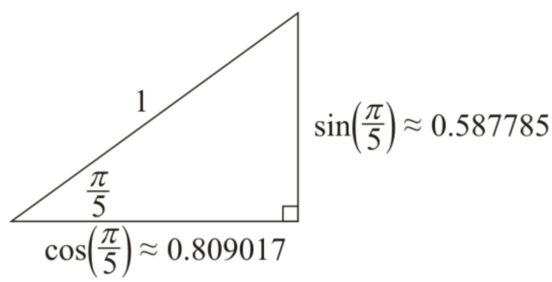


Figure 5. The geometric definition of the sine and cosine of an angle.

Example of the sine and cosine of an angle

For example, if the angle is 36° or $\frac{\pi}{5}$ radians, we can estimate the values of both sin(36°) and cos(36°)[1].





Just in case you forgot, there are 360° in a circle, and there are 2π radians in a circle; consequently, there are $\frac{\pi \text{ rad}}{180°} \approx 0.0174533$ radians per degree, and $\frac{180°}{\pi \text{ rad}} \approx 57.2957795$ degrees per radian. Thus, a degree is to a radian as a unit of measure as a minute is to an hour (at least approximately). To convert degrees to radians, multiply by $\frac{\pi \text{ rad}}{180°}$, and the convert radians to degrees, multiply by the reciprocal $\frac{180°}{\pi \text{ rad}}$. In this case,

$$36°\frac{\pi \text{ rad}}{180°} = \frac{36}{180}\pi \text{ rad} = \frac{1}{5}\pi \text{ rad} = \frac{\pi}{5} \text{ rad}.$$

[1] Incidentally, $\cos(36°) = \cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$ and if you define $\phi = \frac{1+\sqrt{5}}{2}$ to be the golden ratio, this equals $\frac{\phi}{2}$.

You will recall that a circle is divided into exactly 360°, but a radian is defined as that angle that creates an arc along the edge of the circle that has a length equal to the radius of the circle, as shown in Figure 6.

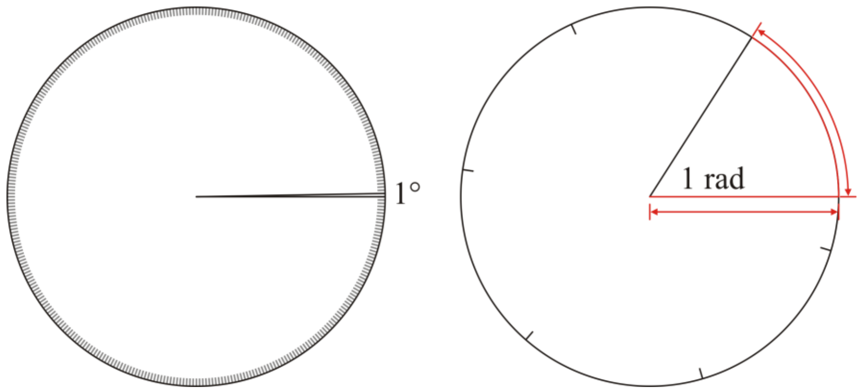


Figure 6. Definitions of the degree and the radian.

It has been an age-old question asking: what is the ratio of the radius of a circle to its circumference? Before the common era, the best approximation was by Claudius Ptolemy, who estimated this ratio as $\frac{377}{60} = 6.28\bar{3}$ (with half of this being $\frac{377}{120} = 3.141\bar{6}$). Today, we know that this ratio is indeed irrational and thus, we simply represent it as 2π; consequently, we say there are 2π radians in a circle, just like there are 360° in a circle. Ptolemy's approximation had a relative error of approximately 0.002% or one part in 42446.5.

Algebraic deductions from trigonometry

Using the Pythagorean theorem, we have that, by definition,

$$\sin^2(\theta)+\cos^2(\theta)=1.$$

Note that we will use the common mathematical notation where $\sin(\theta)\cdot\sin(\theta)=(\sin(\theta))^2=\sin^2(\theta)$.

Incidentally, the Latin word *sinus* referred to the hanging fold of the upper part of a toga, as highlighted in the figure of Claudius on the right (photograph taken by Wikipedia user Sailko). Similarly, the *sine* drops from the intersection of the line and the circle. The word *cosine* comes from the phrase *complementary sine*—that which completes the sine. Reference: OED.

We may also use similar triangles to deduce a few other relationships. First, consider the two similar right-angled triangles shown in Figure 7.

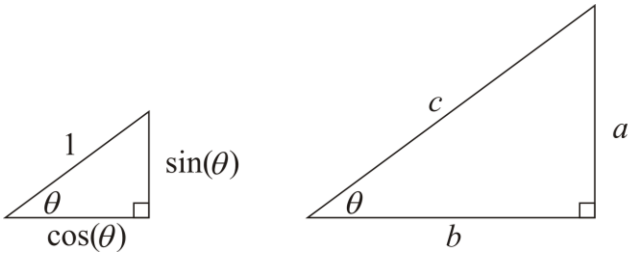


Figure 7. Two similar right-angled triangles.

Using the properties of similar triangles, we may deduce that

$$\frac{a}{c}=\frac{\sin(\theta)}{1}=\sin(\theta),\ \frac{b}{c}=\frac{\cos(\theta)}{1}=\cos(\theta)\ \text{and}\ \frac{a}{c}=\frac{\sin(\theta)}{\cos(\theta)}.$$

The last is usually defined as $\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$. Additionally, we can see that size of the sides relative to the hypotenuse are given by

$$c=\frac{a}{\sin(\theta)}=\frac{b}{\cos(\theta)}.$$

This gives the relationship between the length of the hypotenuse c relative to the opposite side *a* and the adjacent side *b*.

This is where many instructors will, unfortunately, end the geometric interpretation and instead simply define $\cot(\theta)=\frac{1}{\tan(\theta)}$, $\sec(\theta)=\frac{1}{\cos(\theta)}$ and $\csc(\theta)=\frac{1}{\sin(\theta)}$. While these definitions are sufficient to deduce all subsequent properties and relations, it seems somewhat artificial and there is no insight to understanding the significance of identity that $\tan^2(\theta)+1=\sec^2(\theta)$, apart from the rather tedious proof that

$$
\begin{aligned}
\tan^2(\theta)+1 &= \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2 +1 \\
&= \frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} \\
&= \frac{1}{\cos^2(\theta)} \cancelto{1}{\left(\sin^2(\theta)+\cos^2(\theta)\right)} \\
&= \left(\frac{1}{\cos(\theta)}\right)^2 \\
&= \sec^2(\theta)
\end{aligned}
$$

Thus, $\tan^2(\theta)+1=\sec^2(\theta)$. ■

However, the proof uses nothing but substitutions and simple algebra. It leaves one disappointed, as all one has demonstrated is the ability to do algebra and the appropriate substitutions at the appropriate time.

Geometric definitions from trigonometry

Rather, let us not abandon our geometric interpretation just yet: if you draw a line *tangent*[2] to the unit circle at the point $(\cos(\theta), \sin(\theta))$ (the point where the line defined by the angle θ intersects the circle) and extend the tangent to intersect with the *x*-axis, we now have a second triangle, as is shown on the right-hand side of Figure 8. We will define the length of that tangent line to be $\tan(\theta)$. A *secant*[3] is a line that cuts a curve twice, and thus we will complete the triangle by adding the line connecting the origin and where the tangent line cuts the *x*-axis. We will call that line $\sec(\theta)$.

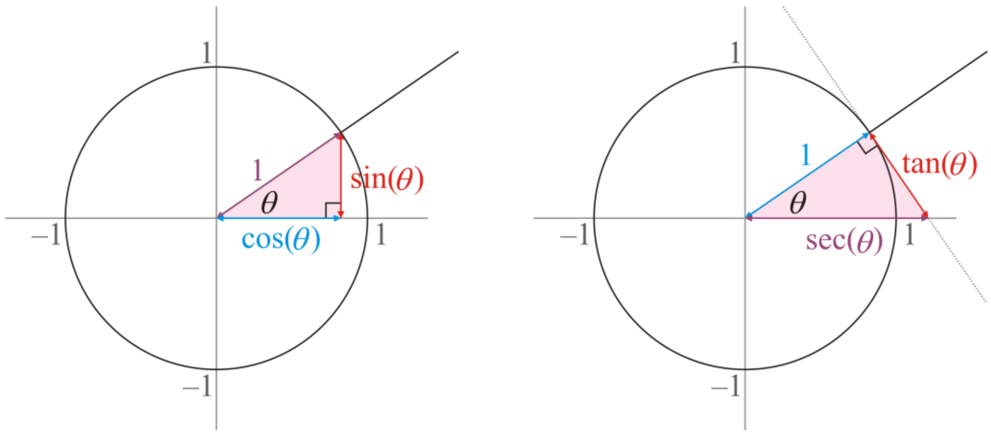


Figure 8. The geometric interpretations of the tangent and secant of an angle θ.

The first observation is that the triangle with sides 1, $\tan(\theta)$ and $\sec(\theta)$ is a right-angled triangle, and thus the Pythagorean theorem applies:

$$1 + \tan^2(\theta) = \sec^2(\theta).$$

Additionally, because the two triangles highlighted in pink have three equal angles, they are similar (the corresponding similar sides are shown in the same color), and thus we may use the properties of similar triangles:

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{\tan(\theta)}{1}, \quad \frac{1}{\cos(\theta)} = \frac{\sec(\theta)}{1} \quad \text{and} \quad \frac{\sin(\theta)}{1} = \frac{\tan(\theta)}{\sec(\theta)}.$$

This gives the familiar formulas of:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \text{ and } \sec(\theta) = \frac{1}{\cos(\theta)},$$

but more importantly, it shows why these relationships are true.

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[2] The term *tangent* originates from the Latin word *tangere*, which means *to touch*. The tangent line "touches" the circle at exactly one point. Reference: OED.

[3] The term *secant* originates from the Latin word *secare*, which means *to cut*. The secant line "cuts" through the circle and extends to the tangent line. Reference: OED.

Another way of visualizing the relationship between tangent, secant, sine and cosine is to draw a tangent line at (1, 0) and extending the line formed by the angle θ until it crosses this tangent line, as shown in Figure 9.

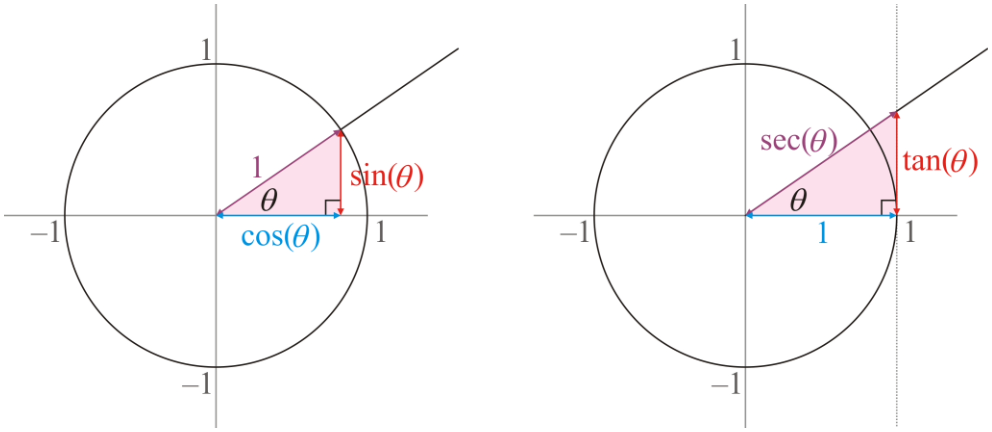


Figure 9. A second interpretations of the tangent and secant of an angle θ.

Again, we see that these two triangles are also similar, and the second triangle is a reflection of the triangle on the right-hand side of Figure 8.

We may also use similar triangles to deduce a few other relationships. First, consider the two similar right-angled triangles shown in Figure 7.

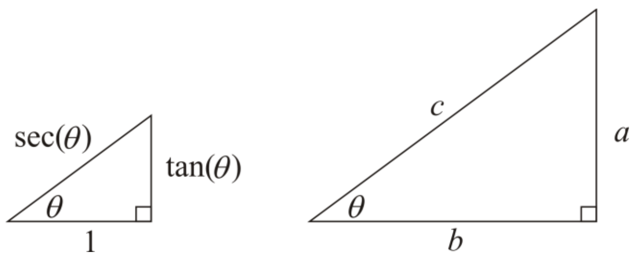


Figure 10. Two similar right-angled triangles.

Using the properties of similar triangles, we may deduce that

$$\frac{a}{b} = \frac{\tan(\theta)}{1} = \tan(\theta), \ \frac{c}{b} = \frac{\sec(\theta)}{1} = \sec(\theta) \text{ and } \frac{a}{c} = \frac{\tan(\theta)}{\sec(\theta)}\left(= \sin(\theta) \text{ from above}\right).$$

Combining these, we get that

$$b = \frac{a}{\tan(\theta)} = \frac{c}{\sec(\theta)}.$$

This gives the relationship between the length of the adjacent side b relative to the opposite side a and the hypotenuse c.

Next, if we consider the other triangle created by the tangent line, as shown in the right-hand image in Figure 11, we have another triangle that is similar to the first. We call the line between the tangent point and the intersection of the line and the y-axis the *complementary tangent* (or *cotangent*, or *cot* for short) of the angle θ. The hypotenuse is called the *complementary secant* (or *cosecant* or *csc* for short).

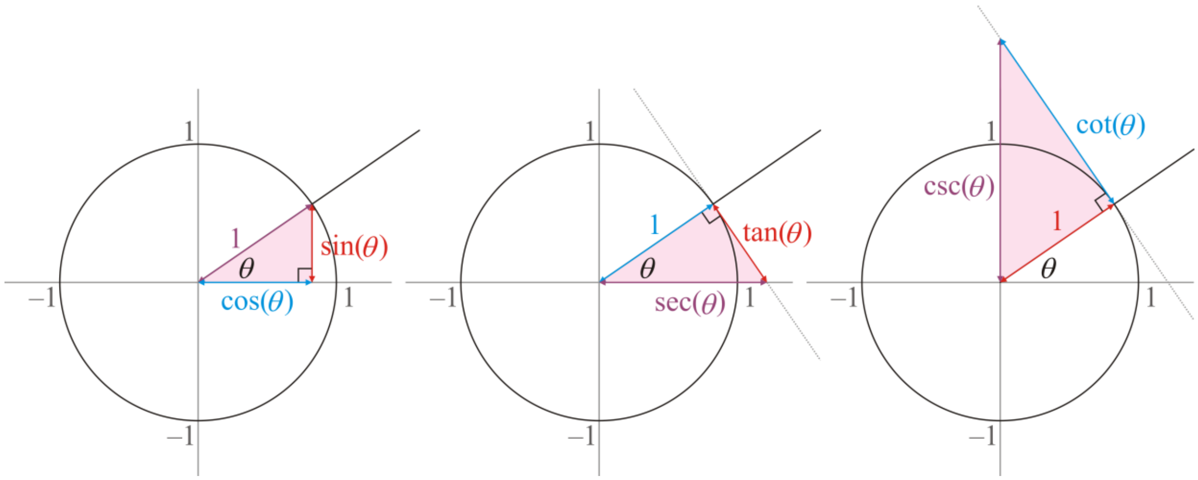


Figure 11. The geometric interpretations of the tangent and secant of an angle θ.

The first observation is that the triangle with sides 1, $\cot(\theta)$ and $\csc(\theta)$ is a right-angled triangle, and thus the Pythagorean theorem applies:

$$1+\cot^2(\theta)=\csc^2(\theta).$$

Additionally, because the two triangles highlighted in pink have three equal angles, they are similar, and thus we may use the properties of similar triangles:

$$\frac{\cos(\theta)}{\sin(\theta)}=\frac{\cot(\theta)}{1},\quad \frac{1}{\sin(\theta)}=\frac{\csc(\theta)}{1}\quad\text{and}\quad \frac{\cos(\theta)}{1}=\frac{\cot(\theta)}{\csc(\theta)}.$$

This gives the familiar formulas of

$$\cot(\theta)=\frac{\cos(\theta)}{\sin(\theta)}\text{ and }\csc(\theta)=\frac{1}{\sin(\theta)}.$$

We can, however, also look at the relationships between the second and third triangles in Figure 11 to see that

$$\frac{1}{\tan(\theta)}=\frac{\cot(\theta)}{1},\quad \frac{\sec(\theta)}{\tan(\theta)}=\frac{\csc(\theta)}{1}\quad\text{and}\quad \frac{\sec(\theta)}{1}=\frac{\csc(\theta)}{\cot(\theta)}.$$

Of these, we have only one additional formula that you may already be familiar with, namely

$$\cot(\theta)=\frac{1}{\tan(\theta)}.$$

Another way of visualizing the relationship between tangent secant, sine and cosine is to draw a tangent line at (0, 1) and extending the line formed by the angle θ until it crosses this tangent line, as shown in

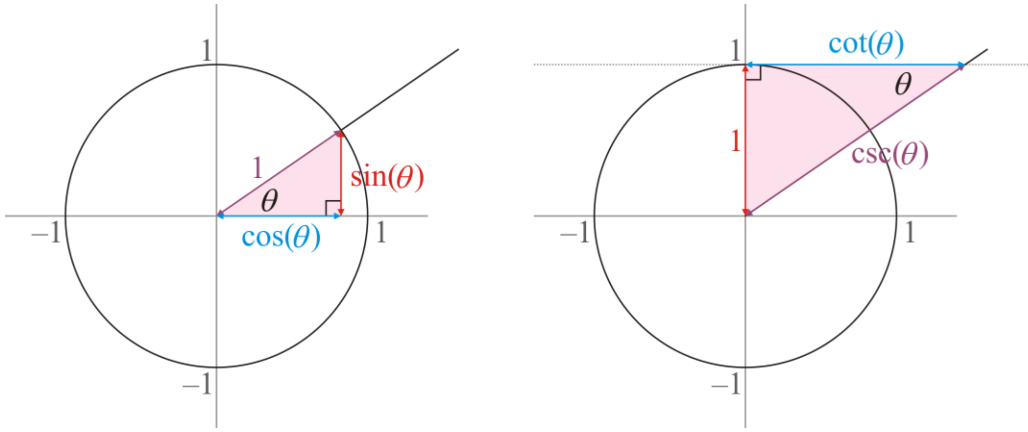


Figure 12. A second interpretations of the complementary tangent and complementary secant of an angle θ.

We may also use similar triangles to deduce a few other relationships. First, consider the two similar right-angled triangles shown in Figure 7.

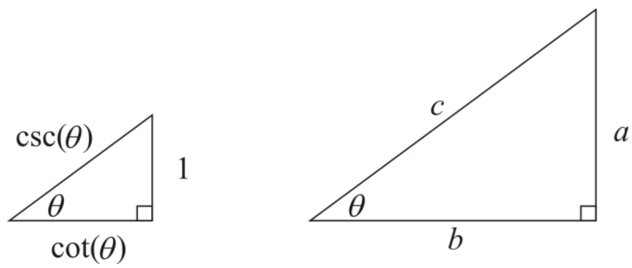


Figure 13. Two similar right-angled triangles.

Using the properties of similar triangles, we may deduce that

$$\frac{b}{a} = \frac{\cot(\theta)}{1} = \cot(\theta),\ \frac{c}{a} = \frac{\csc(\theta)}{1} = \csc(\theta) \text{ and } \frac{b}{c} = \frac{\cot(\theta)}{\csc(\theta)}\left(= \cos(\theta) \text{ from above}\right).$$

Combining these, we get that

$$a = \frac{b}{\cot(\theta)} = \frac{c}{\csc(\theta)}.$$

This gives the relationship between the length of the opposite side a relative to the adjacent side b and the hypotenuse c.

The two images shown in Figure 11 can be summarized in a single image as shown in Figure 14.

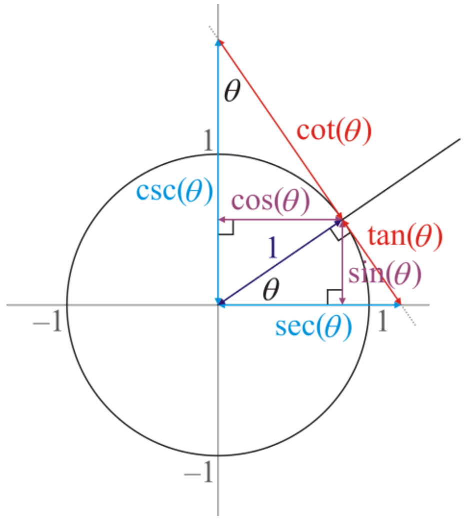


Figure 14. A demonstration of all trigonometric definitions in a single diagram.

In this diagram, we can see six separate triangles that are similar, as highlighted in Figure 15.

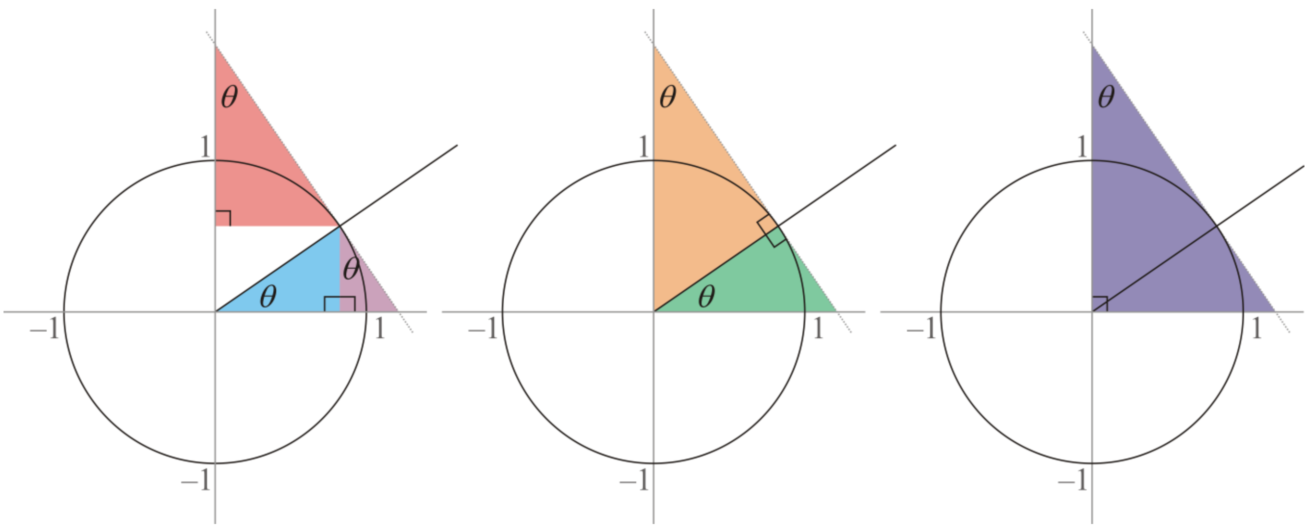


Figure 15. Six similar triangles.

These six together with the length of their sides are listed in Figure 16, in some cases rotated or reflected in order to emphasize the similarity.

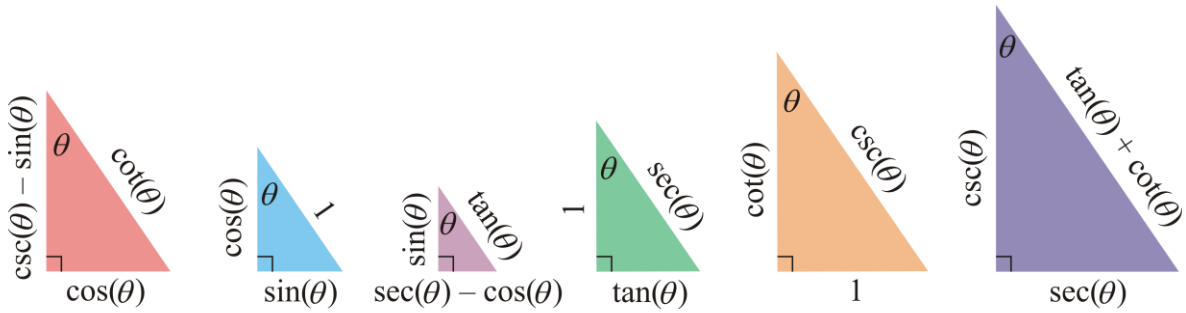


Figure 16. The six similar triangles.

We can therefore apply the Pythagorean theorem to the three additional triangles (the first, third and last in Figure 16):

$$\cos^2(\theta)+(\csc(\theta)-\sin(\theta))^2=\cot^2(\theta),$$

$$(\sec(\theta)-\cos(\theta))^2+\sin^2(\theta)=\tan^2(\theta), \text{ and}$$

$$\sec^2(\theta)+\csc^2(\theta)=(\tan(\theta)+\cot(\theta))^2,$$

respectively. Using the properties of similar triangles, we may now deduce that the following are equal:

$$\frac{\csc(\theta)-\sin(\theta)}{\cot(\theta)}=\frac{\cos(\theta)}{1}=\frac{\sin(\theta)}{\tan(\theta)}=\frac{1}{\sec(\theta)}=\frac{\cot(\theta)}{\csc(\theta)}=\frac{\csc(\theta)}{\tan(\theta)+\cot(\theta)},$$

$$\frac{\cos(\theta)}{\cot(\theta)}=\frac{\sin(\theta)}{1}=\frac{\sec(\theta)-\cos(\theta)}{\tan(\theta)}=\frac{\tan(\theta)}{\sec(\theta)}=\frac{1}{\csc(\theta)}=\frac{\sec(\theta)}{\tan(\theta)+\cot(\theta)}, \text{ and}$$

$$\frac{\cos(\theta)}{\csc(\theta)-\sin(\theta)}=\frac{\sin(\theta)}{\cos(\theta)}=\frac{\sec(\theta)-\cos(\theta)}{\sin(\theta)}=\frac{\tan(\theta)}{1}=\frac{1}{\cot(\theta)}=\frac{\sec(\theta)}{\csc(\theta)}.$$

Those relationships that are likely not familiar to most readers include

$$\cos(\theta)=\frac{\csc(\theta)-\sin(\theta)}{\cot(\theta)}=\frac{\csc(\theta)}{\tan(\theta)+\cot(\theta)},$$

$$\sin(\theta)=\frac{\sec(\theta)-\cos(\theta)}{\tan(\theta)}=\frac{\sec(\theta)}{\tan(\theta)+\cot(\theta)}, \text{ and}$$

$$\tan(\theta)=\frac{\cos(\theta)}{\csc(\theta)-\sin(\theta)}=\frac{\sec(\theta)-\cos(\theta)}{\sin(\theta)},$$

along with reciprocal relationships for $\sec(\theta)$, $\csc(\theta)$ and $\cot(\theta)$, respectively.

Summary of geometric interpretations of trigonometric functions

We have now demonstrated that there are reasonable geometric interpretations of all of the trigonometric functions, and based on the Pythagorean theorem and the properties of similar triangles, we are able to deduce numerous identities from these definitions. **Please don't memorize these formulas**. The purpose of this demonstration is to see what can be easily deduced from this geometric interpretation. Remember the rules that gave us the formulas, not the formulas themselves; being able to instantly regurgitate that $\csc^2(\theta) = 1 + \cot^2(\theta)$ and that $\sec^2(\theta) = 1 + \tan^2(\theta)$ may help you on a test, but it doesn't do much in either in comprehension or in life.

Other formulas

By the very definition of sine, cosine, tangent, cotangent, secant and cosecant, it becomes obvious that in all cases, a very simple relationship holds:

$$\begin{aligned} \sin\left(\frac{\pi}{2} - \theta\right) &= \cos(\theta) & \cos\left(\frac{\pi}{2} - \theta\right) &= \sin(\theta) \\ \tan\left(\frac{\pi}{2} - \theta\right) &= \cot(\theta) & \cot\left(\frac{\pi}{2} - \theta\right) &= \tan(\theta) \\ \sec\left(\frac{\pi}{2} - \theta\right) &= \csc(\theta) & \csc\left(\frac{\pi}{2} - \theta\right) &= \sec(\theta) \end{aligned}$$

Trigonometric functions evaluated at 0°, 30°, 45°, 60° and 90°

Many students easily forget the formulas for simple angles, after all, there does not appear to be a pattern that is easy to remember, for example,

$$\begin{aligned}
\sin(0°) &= \sin(0) = 0 \\
\sin(30°) &= \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \\
\sin(45°) &= \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}. \\
\sin(60°) &= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \\
\sin(90°) &= \sin\left(\frac{\pi}{2}\right) = 1
\end{aligned}$$

The pattern, however, can be observed as follows: consider a progression of $\frac{\sqrt{n}}{2}$ for n = 0, 1, 2, 3, 4:

$$\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}.$$

You will see that in some cases, the square root can be simplified:

$$\frac{0}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{2}{2}.$$

We can then further simplify three more of these ratios:

$$0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1,$$

which are the five values you see above. Thus, as long as you can quickly simplify the above five ratios, it is quite straight-forward to remember the trigonometric formulas for 0°, 30°, 45°, 60° and 90°, as shown in .

Table 1. Trigonometric functions evaluated at common angles.

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\theta)$	$\frac{\sqrt{0}}{2}=0$	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}=1$
$\cos(\theta)$	$\frac{\sqrt{4}}{2}=1$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{0}}{2}=0$
$\tan(\theta)$	$\frac{\sqrt{0}}{\sqrt{4}}=0$	$\frac{\sqrt{1}}{\sqrt{3}}=\frac{1}{\sqrt{3}}$	$\frac{\sqrt{2}}{\sqrt{2}}=1$	$\frac{\sqrt{3}}{\sqrt{1}}=\sqrt{3}$	$\frac{\sqrt{4}}{\sqrt{0}}=\infty$
$\cot(\theta)$	$\frac{\sqrt{4}}{\sqrt{0}}=\infty$	$\frac{\sqrt{3}}{\sqrt{1}}=\sqrt{3}$	$\frac{\sqrt{2}}{\sqrt{2}}=1$	$\frac{\sqrt{1}}{\sqrt{3}}=\frac{1}{\sqrt{3}}$	$\frac{\sqrt{0}}{\sqrt{4}}=0$
$\sec(\theta)$	$\frac{2}{\sqrt{4}}=1$	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{2}}=\sqrt{2}$	$\frac{2}{\sqrt{1}}=2$	$\frac{2}{\sqrt{0}}=\infty$
$\csc(\theta)$	$\frac{2}{\sqrt{0}}=\infty$	$\frac{2}{\sqrt{1}}=2$	$\frac{2}{\sqrt{2}}=\sqrt{2}$	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{4}}=1$

Again, you don't need to memorize these numbers: you should remember the pattern $\frac{\sqrt{0}}{2},\frac{\sqrt{1}}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{3}}{2},\frac{\sqrt{4}}{2}$.

Why radians instead of degrees?

The Babylonians divided a circle into 360 degrees, possibly for two reasons:

1. 360 has many divisors: $360 = 2^3 \cdot 3^2 \cdot 5$, and
2. 360 is very close to the number of days in a year.

The Babylonians also had a base-60 numbering system, although it was more of a hybrid alternating between base 10 and base 6. The Romans adopted this by dividing an hour into 60 primary components (or *minuta prima*), each of which is divided into 60 secondary components (or *minuta secunda*). From this we get the words "minutes" and "seconds" (why we did not adopt "primes" and "seconds" is likely lost in the annals of history).

Thus, there is no real significance to a circle having 360 degrees, each being divided into 60 minutes of arc, and each minute of arc (or *arcminute*) being divided into 60 seconds of arc (or *arcsecond*).

Indeed, the directions on the compass do not even work well with degrees:

1. West is 90° to the left of North,
2. NW is 45° to the left of North,
3. NNW is 22.5° to the left of North, and
4. NbW is 12.25° to the left of North.

The military uses an alternative unit, the *mil*, where a circle is divided into 6400 mils. In some sense, this is more convenient, as now:

1. West is 1600 mils to the left of North,
2. NW is 800 mils to the left of North,
3. NNW is 400 mils to the left of North, and
4. NbW is 200 mils to the left of North.

There is, however, one significant benefit of mils over degrees: if you aim at a target but are off by one mil on either side and your target is one kilometre away, your bullet will be off by very close to one metre in the direction of error, and if you are off by 7 mils, you will miss the target very close to seven metres. On the other hand, if you are off by

1. one degree, you will miss that same target by 17.45 m,
2. one minute, you will miss that target by 29 cm, and
3. one second, you will miss the target by 5 mm.

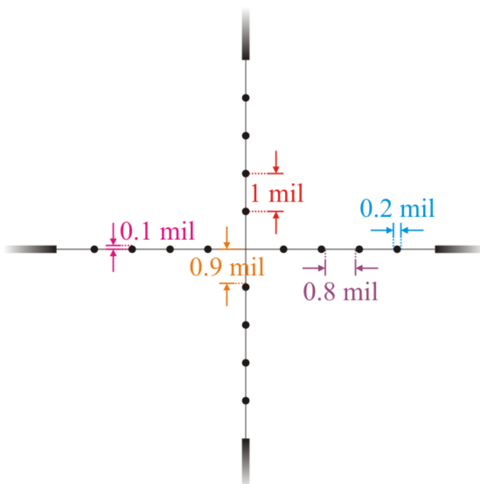
As you may guess, the military approach can most easily be calculated in one's head, and soldiers have a solid grasp of mils before they graduate from boot camp. What's odd, however, is that engineering students tend to prefer degrees, because that's what they learned in elementary and secondary school, and because, unlike soldiers (who are drilled to simply accept what is required of them), they desperately try to hold onto to what they originally learned.

Instead, let's look at that interesting relationship: $\tan(1\text{ mil}) \approx \frac{1\text{ m}}{1\text{ km}} = \frac{1\text{ m}}{1000\text{ m}} = 0.001$. In this case, it seems that $1\text{ mil} \approx 0.001\text{ rad}$, and if you remember your metric prefixes, $0.001\text{ rad} = 1\text{ mrad}$ or one *milliradian*. Thus, one mil is almost exactly equal to one milliradian. It is not precise, though, for there are 6400 mils in a circle while there are $2000\pi\text{ mrad} \approx 6283.1853\text{ mrad}$ per circle. Thus, one mil is only approximately equal to a milliradian, but for the applications where they are used, the approximation that $\tan(1\text{ mil}) \approx 0.001$ is good enough.

For the mil as an approximation to a milliradian, the *relative error* is quite small:

$$\frac{\left|\frac{1}{6400}-\frac{1}{2\pi\times 1000}\right|}{\frac{1}{2\pi\times 1000}}\times 100\ \% \approx 1.85\ \% \ .$$

A sniper scope has engraved in it a *reticle* with a cross hair and sixteen dots, with each dot being exactly one milliradian away from the next along horizontal and vertical hairs. These are exactly one milliradian, and not one mil, as for a sniper, an error of 1.85 % is actually very significant. The dots themselves are 0.2 mrad in diameter and the cross hairs are 0.02 mrad in thickness.



Consequently, some engineering students seem to dislike radians simply on principle, and yet every soldier going through boot camp learns how to use radians, only a simplified version of radians where $2\pi \approx 6.4$. Perhaps this approximation is not as good as Ptolemy's approximation (recall that Ptolemy used the approximation of $\frac{377}{60}=6.28\bar{3}$), but it is still useful. Consequently, while radians are feared by some engineering students, perhaps it easiest to just remember that a radian is approximately 57.3° and get over it.